Factorial: Group Exercises

CSCI 246

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Problem 1. Suppose you have n books and n shelves on your bookshelf. How many ways can you arrange the books so that each shelf has exactly one book?

A. $n = 1$: $1! = 1$

B. $n = 2$: $2! = 2$

C. $n = 3$: $3! = 6$

D. $n = 4$: $4! = 24$

E. $n = 5$: $5! = 120$

F. $n = 6$: $6! = 720$

Problem 2.

A. Give a relationship between orderings of lists and factorials.

The number of permutations of a list of length n is $n!$. Or alternatively, the number of lists of length n from n elements with no repeats is $(n)_n = n!$

B. Define $n!$ in terms of a product (e.g., $n! = \Pi_{k=\ldots}^{\ldots} e_k$)

$$n! = n(n-1)(n-2)\ldots 1 = \prod_{k=1}^{n} k$$

Problem 3. Give an explanation *in terms of counting* why $(n)_k = \frac{n!}{(n-k)!}$.

The number of lists of length k with no repeats from n distinct elements is $(n)_k$. Similarly, the number of lists of length n with no repeats from n distinc elements is $n!$.

One way to count the number of lists of length k with no repeats from n elements, is to first count the number of lists with no repeats that uses all n elements (i.e., $n!$ total lists). However, we are only concerned with how many unique prefixes of length k there are. Thus, we wish to ignore the last $(n-k)$ elements of the n length lists. For each unique k length prefix, there are a total of $(n-k)!$ ways to complete the list. Hence, why there are a total of $(n)_k = \frac{n!}{(n-k)!}$ unique k length prefixes of the n-length lists with no repeats.

Problem 4. Let $A = \{a, b, c, d, e\}$ be a set of elements.

A. How many lists of length 5 can be made with elements of A? $5^5 = 3125$

B. How many lists of length 5 with no repeats? $5! = 120$

C. How many subsets of A have size 5? 1

D. How many lists of length 2? $5^2 = 25$

E. How many lists of length 2 with no repeats? $(5)_2 = 5 * 4 = 20$

F. How many subsets of size 2? $(5)_2/2! = 5 * 4/2 = 10.$

G. How many lists of length 3 that begin with a? $1 * 5^2 = 25$

H. How many lists of length 3 with no repeats that begin with a? $1 * (4)_2 = 1 * 4 * 3 = 12.$

I. How many subsets of size 3 that contain a? $1 * (4)_2/2! = 1 * 4 * 3/2 = 6$

Problem 5. Suppose you have 5 French books, 3 Spanish Books, 4 English Books, and 12 German books and you want to order them on your book shelf.

A. How many ways can you order the books?

$(5 + 3 + 4 + 12)! = 24! = 620448401733239439360000$

B. How many ways if you want to keep books organized by language?

(number of ways to order languages) $\times$ (number of ways to order books within each language)
$4! \times (5! \times 3! \times 4! \times 12!) = 24 \times (120 \times 6 \times 24 \times 479001600) = 198651543552000$

Problem 6. Make a conjecture to the value of the following summation:

$$\sum_{k=1}^{n} k \times k! = \sum_{k=1}^{n} (k + 1 - 1) \times k! = \sum_{k=1}^{n} (k + 1)k! - k! = \sum_{k=1}^{n} (k + 1)! - k!$$

$$\sum_{k=1}^{n} (k + 1)! - k! = (2! - 1!) + (3! - 2!) + ...((n + 1)! - n!) = (n + 1)! - 1$$

Problem 7. Give an argument why for any integers k and n, if $2 \leq k \leq n$, then $n! + k$ is composite.

Proof. Note, by definition $n! = 1 \times 2 \times 3 \times \cdots \times n$. Since $2 \leq k \leq n$, we know that k must be one of the factors of $n!$ (i.e, $k|n!$). Necessarily, there must be some integer m such that $n! = k \times m$. Now consider, $n! + k = k \times m + k = k(m + 1)$. Clearly, $k|(n! + k)$. Necesarrily, $1 < k < n! + k$. Thus, $n! + k$ is composite. $\square$