Quantifiers: Group Exercises

CSCI 246

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Problem 1. Rewrite the following statements using $\exists$ and/or $\forall$ quantifier symbols. You may also use symbols for boolean connectives (e.g., $\implies$ for if-then statements, $\iff$, $\neg$ for negation, $\vee$ for disjunction, and $\wedge$ for conjunction). **Note**: Not all of the following statements are true.

A. Any integer is prime or composite.

Let $\mathit{prime}(x) \triangleq x$ is prime and $\mathit{composite}(x) \triangleq x$ is composite.

$$\forall x \in \mathbb{Z}.\ \mathit{prime}(x) \vee \mathit{composite}(x)$$

B. There is a natural number greater than all other natural numbers.

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ n \geq m$$

C. Every integer divisible by 9 and 2 is divisible by 18.

$$\forall x \in \{y \in \mathbb{Z} : 2|y \wedge 9|y\}.\ 18|x \qquad \text{or} \qquad \forall x \in \mathbb{Z}.\ (2|x \wedge 9|x) \Rightarrow 18|x$$

D. For every integer x, there is a prime number greater than x.

Let $\mathit{prime}(x) \triangleq x$ is prime.

$$\forall x \in \mathbb{Z}.\ \exists p \in \{y \in \mathbb{Z} : \mathit{prime}(y)\}.\ p \geq x \qquad \text{or} \qquad \forall x \in \mathbb{Z}.\ \exists p \in \mathbb{Z}.\ \mathit{prime}(p) \wedge p \geq x$$

E. For each prime number p, there exists a prime number less than p.

Let $\mathit{prime}(x) \triangleq x$ is prime.

$$\forall p \in \mathbb{Z}.\ \mathit{prime}(p) \Rightarrow \exists p' \in \mathbb{Z}.\ \mathit{prime}(p') \wedge p' < p.$$

Problem 2. Translate each symbolic statement to English. **Note**: not all statements will be true.

A. $\forall n \in \mathbb{Z}.\ \exists m \in \mathbb{Z}.\ m = n + 1.$

For each integer, there is another one more than it.

B. $\exists n \in \mathbb{Z}.\ \forall m \in \mathbb{Z}.\ m \geq n.$

There is an integer, less than every other integer.

C. $\forall n \in \mathbb{Z}.\ 2 < n \wedge \mathit{prime}(n) \implies \mathit{odd}(n).$

Every prime number greater than 2 is odd.

D. $\exists n \in \mathbb{Z}.\ \mathit{prime}(n) \wedge \mathit{even}(n).$

There is an integer that is prime and even.

E. $\forall n \in \mathbb{Z}.\ \mathit{composite}(n) \implies \exists k \in \mathbb{Z}.\ 1 < k < n \wedge k|n.$

Each composite number has a divisor strictly between 1 and itself.

Problem 3. Determine whether the following statements are true or false. Provide a brief justification for your answer.

A. $\forall n \in \mathbb{Z}.\ \exists m \in \mathbb{Z}.\ m > n \wedge n|m$.

The statement is true. Let $m = 2n$, clearly $2n > m$ and $n|2n$.

B. $\exists n \in \mathbb{Z}.\ \forall m \in \mathbb{Z}.\ n|m$.

The statement can be true depending on if your definition of divisibility allows for $0|0$. If yes, then let $n = 0$. Otherwise, if not $0|0$ then the statement is false.

C. $\forall n \in \mathbb{Z}.\ n > 1 \implies \exists p \in \mathbb{Z}.\ \mathit{prime}(p) \wedge p|n$.

The statement is true. Either n and prime and let $p = n$. Or n is compisite, and there is some prime divisor of n we can instantiate p with.

D. $\exists n \in \mathbb{Z}.\ \forall m \in \mathbb{Z}.\ m < n$.

The statement is false. There is no natural number greater than all natural numbers.

E. $\forall m \in \mathbb{Z}.\ \exists n \in \mathbb{Z}.\ m < n$.

The statement is true. Let $n = m + 1$. Clearly $m < m + 1$.

Problem 4. Prove $\exists x \in \mathbb{Z}.\ \mathit{even}(x) \wedge \mathit{prime}(x)$.

Proof. Let $x = 2$. Clearly 2 is even and 2 is prime. $\square$

Problem 5. Prove $\forall n \in \mathbb{Z}.\ n^2 \geq 0$.

Proof. Let n be an arbitrary integer. Either $n < 0$, $n = 0$, or $0 < n$. In the first case, a negative times a negative is positive, thus $n^2 \geq 0$. In the second case, $n^2 = 0$ so clearly $n^2 \geq 0$. And in the final case, a positive times a positive is positive, thus $n^2 \geq 0$. In all cases, we have $n^2 \geq 0$. $\square$

Problem 6. Prove $\forall n \in \mathbb{N}.\ n > 1 \implies \exists p \in \mathbb{Z}.\ \mathit{prime}(p) \wedge p|n$. You may assume that every *composite* number has at least one unique *prime* divisor.

Proof. Let n be any natural number. We must prove if $n > 1$, then there is a prime number that divides n.

Since $n > 1$, either n is prime or n is composite.

In the case that n is prime, let $p = n$. Clearly n is prime and $n|n$.

In the case that n is composite, we know that n has at least one prime factor. Lets p, be one such prime factor of n. Since p is a prime factor of n, we know that p is prime and $p|n$ as required. $\square$