Relations: Group Exercises

CSCI 246

February 13, 2026

Problem 1. Let $A = \{1, 2, 3, 4, 5\}$ write down the following relations on A.

A. The *is-less-than* relation.

$$\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$$

B. The *is-divisible-by* relation.

$$\{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5)\}$$

C. The *is-equal-to* relation.

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$$

D. The *has-same-parity* relation.

$$\{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$$

Problem 2. Each of the following are relations on the set $A = \{1, 2, 3, 4, 5\}$. Describe each in English.

A. $\{(1, 2), (2, 3), (3, 4), (4, 5)\}$

The relation *is-one-less-than*, i.e., $xRy \iff x = y - 1$.

B. $\{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$.

The relation *greater-than-or-equal-to*, i.e., $xRy \iff x \geq y$.

C. $\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$.

The relation *sums-to-six*, i.e., $xRy \iff x + y = 6$.

D. $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)\}$.

The relation *divides*, i.e., $xRy \iff x|y$.

Problem 3. For each relation on the set $A = \{1, 2, 3, 4, 5\}$, determine if the relation is reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

A. $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

The relation R is reflexive, symmetric, and antisymmetric.

B. $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$

The relation R is irreflexive and antisymmetric.

C. $R = \{(1, 1), (1, 2), (2, 1), (3, 4), (4, 3)\}$

The relation R is symmetric.

D. $R = A \times A$

The relation R is reflexive, symmetric, and transitive.

Problem 4. For each relation R, provide the inverse relation R^{-1}.

A. $R = \{(1,2),(2,3),(3,4)\}$

$$R^{-1} = \{(2,1),(3,2),(4,3)\}$$

B. $R = \{(1,1),(2,2),(3,3)\}$

$$R^{-1} = \{(1,1),(2,2),(3,3)\}$$

C. $R = \{(x,y) : x,y \in \mathbb{Z}.\ x-y=1\}$

$$R^{-1} = \{(y,x) : x, y\ in\mathbb{Z}.\ x-y=1\} \text{ or alternatively } R^{-1} = \{(x,y) : x,y \in \mathbb{Z}.\ y-x=1\}$$

D. $R = \{(x,y) : x,y \in \mathbb{N}.\ x|y\}$

$$R^{-1} = \{(y,x) : x, y\ in\mathbb{N}.\ x|y\} \text{ or alternatively } R^{-1} = \{(x,y) : x,y \in \mathbb{N}.\ y|x\}$$

E. $R = \{(x,y) : x,y \in \mathbb{Z}.\ xy > 0\}$

$$R^{-1} = R = \{(x,y) : x,y \in \mathbb{Z}.\ xy > 0\} = \{(y,x) : x,y \in \mathbb{Z}.\ xy > 0\} = \{(x,y) : x,y \in \mathbb{Z}.\ yx > 0\}$$

Problem 5. Let R and S be relations such that $R = S^{-1}$. Prove that $S = R^{-1}$.

Proof. Let R and S be relations such that $R = S^{-1}$.

By definition of set equality, we know that for arbitrary x and y, $xRy \iff xS^{-1}y$.

Similarly, by definition of *inverse*, we know that for any x and y that $xRy \iff yR^{-1}x$ and $ySx \iff xS^{-1}y$.

Combining these facts, we have for any x and y:

$$ySx \iff xS^{-1}y \iff xRy \iff yR^{-1}x$$

Simplifying, we have for any x and y, $ySx \iff yR^{-1}x$, and thus $S = R^{-1}$.

□

Problem 6. Prove that a relation R is symmetric if and only if $R = R^{-1}$.

Proof.

Case $\Rightarrow$: if R is symmetric, then $R = R^{-1}$.

Let R be a symmetric relation.
Let x and y be any R-related elements (i.e., xRy).
Since R is symmetric, it must be the case that yRx.
By definition of inverse, since yRx it must be the case that $xR^{-1}y$.
Since x and y were arbitrarily chosen, we may conclude for any x and y, $xRy \iff xR^{-1}y$.
Thus, $R = R^{-1}$.

Case $\Leftarrow$: If $R = R^{-1}$, then R is symmetric.

Let R be any relation such that $R = R^{-1}$.
Let x and y be any R-related elements (i.e., xRy).
By definition of inverse, since xRy it must be the case that $yR^{-1}x$.
Since $R = R^{-1}$, we have yRx.
Since x and y were arbitrarily chosen, we may conclude that for any x and y, $xRy \iff yRx$.
Thus, we may finally conclude that R is symmetric.

Since we have proved both directions, we may conclude that R is symmetric if and only if $R = R^{-1}$. □

Problem 7. Prove that a relation $R \subseteq A \times A$ is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$.

Proof.

Case $\Rightarrow$: If $R \subseteq A \times A$ is antisymmetric, then $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$

Let $R \subseteq A \times A$ be a relation on A that is antisymmetric.

Let x and y be any elements related by the relation $R \cap R^{-1}$; i.e., any pair such that xRy and $xR^{-1}y$.

Since $xR^{-1}y$, it must be the case that yRx.

Since xRy, yRx, and R is anti-symmetric we know that $x = y$.

Thus, $(x,y) \in \subseteq \{(a,a) : a \in A\}$.

Since (x,y) was arbitrarily chosen from $R \cap R^{-1}$ and $(x,y) \in \{(a,a) : a \in A\}$, we may conclude that $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$.

Case $\Leftarrow$: If $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$, then R is antisymmetric.

Let $R \subseteq A \times A$ be any relation on A that satisfies $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$.

The relation R is anti-symmetric if and only if for any x and y, if xRy and yRx, then $x = y$.

Let x and y be any elements of A such that xRy and yRx.

Since yRx, we know that $xR^{-1}y$.

Since both xRy and $xR^{-1}y$, we know that $(x,y) \in (R \cap R^{-1})$.

Further, since $(x,y) \in (R \cap R^{-1})$ and $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$, we have $(x,y) \in \{(a,a) : a \in A\}$.

Since $(x,y) \in \{(a,a) : a \in A\}$—which only relates equal terms—it must be that $x = y$.

Since x and y are arbitrary elements such that xRy and yRx, we may conlcude that R is antisymmetric.

Since we have proved both directions, we may conclude that $R \subseteq A \times A$ is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$. $\square$