Equivalence Relations: Group Exercises

CSCI 246

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Problem 1. Find all values of $N > 1$ that make the following congruences true.

A. $23 \equiv 13 \pmod{N}$. $N \in \{k \in \mathbb{N} : k > 1 \wedge k|(23-13)\} = \{2, 5, 10\}$

B. $10 \equiv 5 \pmod{N}$. $N \in \{k \in \mathbb{N} : k > 1 \wedge k|(10-5)\} = \{5\}$

C. $6 \equiv 60 \pmod{N}$. $N \in \{k \in \mathbb{N} : k > 1 \wedge k|(60-6)\} = \{2, 3, 6, 9, 18, 27, 54\}$

D. $23 \equiv 22 \pmod{N}$. $N \in \{k \in \mathbb{N} : k > 1 \wedge k|(23-13)\} = \emptyset$

Problem 2. Let a and b be distinct integers (i.e., $a \neq b$), what is the largest N such that $a \equiv b \pmod{N}$? Explain.

The largest N such that $a \equiv b \pmod{N}$ is $|a-b|$. This comes from the fact that $\equiv \pmod{N}$ is defined as the relation $\{(x, y) \in \mathbb{Z}^2 : N|(x-y)\}$, and the largest factor of $x - y$ is $|x - y|$.

Problem 3. Which of the following relations are equivalence relations?

A. $R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$ on $\{1, 2, 3\}$. Yes

B. $R = \{(1,2), (2,3), (3,1)\}$ on $\{1, 2, 3\}$. No, not reflexive.

C. $|$ on $\mathbb{Z}$. No, not symmetric.

D. $\leq$ on $\mathbb{Z}$. No, not symmetric.

E. $\{1, 2, 3\} \times \{1, 2, 3\}$ on the set $\{1, 2, 3\}$. Yes.

F. $\{1, 2, 3\} \times \{1, 2, 3\}$ on the set $\{1, 2, 3, 4\}$. No, not reflexive.

Problem 4. For each equivalence relation R, find the requested equivalence classes.

A. $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$ on $\{1,2,3,4\}$. Find [1], [2], [3], and [4].

$$[1] = [2] = \{1,2\} \qquad [3] = \{3\} \qquad [4] = \{4\}$$

B. R is *has-same-tens-digit* on the set $\{x : 100 < x < 200\}$. Find [123].

$$[123] = \{120, 121, 122, 123, 124, 125, 126, 127, 128, 129\}$$

C. R is *has-same-cardinality* on the set $\mathcal{P}(\{1,2,3,4,5\})$. Find $[\{1,3\}]$.

$$[\{1,3\}] = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

Problem 5. For each equivalence relation, determine how many unique equivalence classes of R there is.

A. $\equiv \pmod{10}$ on $\mathbb{Z}$. There are 10 equivalence classes of $\equiv \pmod{10}$ on $\mathbb{Z}$.

B. *has-same-parity* on $\mathbb{Z}$. There are 2 equivalence classes of *has-same-parity* on $\mathbb{Z}$..

C. *has-same-cardinality* on $\mathcal{P}(A)$ where $|A| = n$.

There are $n+1$ equivalence classes of *has-same-cardinality* on $\mathcal{P}(A)$ if $|A| = n$.

Problem 6. Prove that for any equivelance relation R on A that the union of all equivalence classes of R is equivalent to A; i.e.,

$$\bigcup_{a \in A} [a] = A$$

Proof.

Case $\Rightarrow$: $x \in \bigcup_{a\in A}[a] \Rightarrow x \in A$.

Let x be any element in $\bigcup_{a\in A}[a]$. Necessarily, there must be some $a \in A$, such that $x \in [a]$. By definition, $[a] = \{x : xRy\}$ and $R \subseteq A \times A$ and thus, $[a] \subseteq A$. Therefore, $x \in A$.

Case $\Leftarrow$: $x \in A \Rightarrow x \in \bigcup_{a\in A}[a]$.

Let $x \in A$ be any element in A. Since $R \subseteq A \times A$ is an equivalence relation, necessarily xRx, and as such $x \in [x]$. Since $x \in A$ and $x \in [x]$, we may conclude that $x \in \bigcup_{a\in A}[a]$.

Since we have proven both $x \in \bigcup_{a\in A}[a] \Rightarrow x \in A$ and $x \in A \Rightarrow x \in \bigcup_{a\in A}[a]$ we may conclude that $\bigcup_{a\in A}[a] = A$. □

Problem 7. Let R and S be any equivalence relations on A. Prove that if R and S have the exact same set of equivalence classes, then $R = S$; i.e.,

$$(\forall a \in A.\ [a]_R = [a]_S) \implies R = S$$

where $[a]_R$ means the equivalence class of a in R and similarly $[a]_S$ the equivalence class of a in S.

Proof.

Let R and S be equivalence relations on A that have the same set of equivalence classes.
By the Lemma in **Problem 6**, we know $\bigcup_{a\in A}[a]_R = R$ and $\bigcup_{a\in A}[a]_S = S$.
By assumption, we know that each $[a]_R$ is equal to $[a]_S$. Thus, $R = \bigcup_{a\in A}[a]_R = \bigcup_{a\in A}[a]_S = S$.
Therefore, $R = S$. □