

Partitions: Group Exercises

CSCI 246

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Problem 1. Which of the following are partitions?

- A. $P = \{\emptyset, \{1, 2, 3\}\}$ on $\{1, 2, 3\}$. Not a partition, contains \emptyset .
- B. $P = \{\{1, 2, 3\}\}$ on $\{1, 2, 3\}$. Is a partition.
- C. $\{\{y \in \mathbb{Z} : y|x\} : x \in \mathbb{Z}\}$ on \mathbb{Z} . Not a partition, contains overlapping subsets of \mathbb{Z} .
- D. $\{\{y \in \mathbb{Z} : y \leq x\} : x \in \mathbb{Z}\}$ on \mathbb{Z} . Not a partition, contains overlapping subsets of \mathbb{Z} .
- E. $\{\{1\}, \{2\}, \{3\}\}$ on the set $\{1, 2, 3\}$. Is a partition.
- F. $\{\{1, 2, 3\}\}$ on the set $\{1, 2, 3\}$. Is a partition; oops same as part B.

Problem 2. List the partition created by the following equivalence relations on their respective sets.

- A. $\equiv \pmod{4}$ on $\{x \in \mathbb{Z} : 0 \leq x \leq 20\}$.
 $\{\{0, 4, 8, 12, 16, 20\}, \{1, 5, 9, 13, 17\}, \{2, 6, 10, 14, 18\}, \{3, 7, 11, 15, 19\}\}$
- B. $\{a, b, c, d\} \times \{a, b, c, d\}$ on $\{a, b, c, d\}$.
 $\{\{a, b, c, d\}\}$
- C. $=$ on $\{1, 2, 3, 4, 5, 6\}$
 $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$
- D. $(\{a, b\} \times \{a, b\}) \cup (\{c, d\} \times \{c, d\})$ on $\{a, b, c, d\}$
 $\{\{a, b\}, \{c, d\}\}$

Problem 3. List all partitions of the set $A = \{1, 2, 3\}$.

- $\{\{1\}, \{2\}, \{3\}\}$ $\{\{1, 2\}, \{3\}\}$ $\{\{1, 3\}, \{2\}\}$ $\{\{1\}, \{2, 3\}\}$ $\{\{1, 2, 3\}\}$

Problem 4. Prove that for every partition P of A , $\overset{P}{\equiv}$ is an equivalence relation on A .

Proof. Let P be any partition on A .

Case: $\overset{P}{\equiv}$ is reflexive.

Let $x \in A$ be any element of A . By definition $\bigcup P = A$.

Necessarily, there must be some $p \in P$ such that $x \in p$.

By definition $\overset{P}{\equiv}$ relates elements within the same part; and thus $x \overset{P}{\equiv} x$.

Case: $\overset{P}{\equiv}$ is symmetric.

Let $x \overset{P}{\equiv} y$ be any elements of A related by $\overset{P}{\equiv}$.

By definition, there is some part $p \in P$ such that $x \in p$ and $y \in p$.

Since x and y are in the same part, we may conclude that $y \overset{P}{\equiv} x$.

Case: $\overset{P}{\equiv}$ is transitive.

Let $x \overset{P}{\equiv} y$ and $y \overset{P}{\equiv} z$.

By definition, we know there are some parts $p, p' \in P$ such that $x, y \in p$ and $y, z \in p'$.

Since P is a partition, we know by definition we know that the parts of P are disjoint.

Since $y \in p$ and $y \in p'$ we know that $p = p'$ and therefore x and z are in the same part.

Since x and z are in the same part of P , we may conclude that $x \overset{P}{\equiv} z$.

Since $\overset{P}{\equiv}$ is a relation on A that is reflexive, symmetric, and transitive, we may conclude that (by definition) $\overset{P}{\equiv}$ is an equivalence relation. □

Problem 5. Prove that for every equivalence relation R on A , $P_R = \{[a]_R : a \in A\}$ is a partition of A .

Proof.

Let R be an equivalence relation on A and let P_R be the set of all equivalence classes of R .

Case $\emptyset \notin P_R$.

By definition P_R is the set containing all equivalence classes of R .

And since each element is an equivalence class and $a \in [a]$ for any $a \in A$, we can conclude $\emptyset \notin P_R$.

Case P_R is pairwise disjoint.

Assume P_R contains some distinct parts p and p' that are not disjoint.

Since p and p' are not disjoint, there must be some $x \in A$ such that $x \in p$ and $x \in p'$.

Additionally, there must be some $a, a' \in A$ such that $[a] = p$ and $[a'] = p'$.

Since $[a]$ and $[a']$ are equivalence classes and $x \in [a]$ and $x \in [a']$, we know $[a] = [a']$.

A contradiction, thus P_R is pairwise disjoint.

Case $\bigcup P_R = A$.

By definition $P_R = \{[a] : a \in A\}$ and thus $\bigcup P_R = \bigcup_{a \in A} [a]$.

Additionally, from the lemma in **Problem 6** of "Equivalence Relations" exercises, $\bigcup_{a \in A} [a] = A$.

Thus $P_R = A$.

Since P_R is a set of subsets of A that does not contain the emptyset, are pairwise disjoint, and covers A , we may conclude that (by definition) P_R is a partition. □

Problem 6. Prove that for every equivalence relation R on A , if $P_R = \{[a]_R : a \in A\}$, then $R = \overset{P_R}{\equiv}$.

Proof. Let R be any equivalence relation on A and P_R be the set of all equivalence classes of R .

Case \Rightarrow : $xRy \Rightarrow x \overset{P_R}{\equiv} y$.

Let xRy be any pair of elements related by the relation R .

Since R is an equivalence relation and xRy , we know that $x \in [y]$ and $y \in [y]$.

By definition P_R is the partition whose parts are the equivalence classes of R .

Thus, $\overset{P_R}{\equiv}$ is a relation that relates elements in the same equivalence class of R .

Therefore, $x \overset{P_R}{\equiv} y$.

Case \Leftarrow : $x \overset{P_R}{\equiv} y \Rightarrow xRy$.

Let $x \overset{P_R}{\equiv} y$ be any pair of elements related by $\overset{P_R}{\equiv}$.

By definition P_R is the partition whose parts are the equivalence classes of R .

Thus, $\overset{P_R}{\equiv}$ is a relation that relates elements in the same equivalence class of R .

Since $x \overset{P_R}{\equiv} y$, we know there is some equivalence class of R that contains both x and y .

Namely, the equivalence class $[y]$. E.g., $x, y \in [y]$. By definition $x \in [y]$ if and only if xRy .

Clearly, xRy .

Since for any $x, y \in A$, we have $xRy \Leftrightarrow x \overset{P_R}{\equiv} y$, we may conclude that $R = \overset{P_R}{\equiv}$.

□