Binomial Coefficients: Group Exercises

CSCI 246

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Problem 1. Compute the first 6 rows for Pascal's Triangle (i.e., $\binom{n}{k}$ for all $0 \leq k \leq n \leq 5$).

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Problem 2. Let $B = \{0,1\}^n$ be the set of all bitstrings of length n.

A. What is $|B|$? 2^n

B. How many elements of B have exactly k bits set? $\binom{n}{k}$

C. How many elements of B have exactly $(n-k)$ bits set (i.e., k bits unset)? $\binom{n}{n-k} = \binom{n}{k}$

D. How many elements of B have at most 3 bits set? $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$

Problem 3. Consider the binomial $(x+y)^6$.

A. What is the coefficient of the x^3 term? 0

B. What is the coefficient of the x^5y term? 6

C. What is the coefficient of the x^3y^3 term? 20

D. What is the coefficient of the y^2 term? 0

Problem 4. Prove for $0 \leq k \leq n$ that:

$$\left(\begin{array}{c} n \\ 2 \end{array} \right) = \sum_{k=1}^{n-1} k$$

Proof.

The left hand side is equal to $\frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!2!} = \frac{n(n-1)}{2}$.

On the right hand side we have $\sum_{k=1}^{n-1} k = 1 + 2 + ... + (n-2) + (n-1)$.

There are a total of $n-1$ terms in the sum, with an average value of $\frac{n}{2}$.

That is consider the the first and last term sum to n, the 2nd and 2nd-to-last sum to n, and so on.

Thus, the right-hand summation is equal to $(n-1)\frac{n}{2} = \frac{n(n-1)}{2}$—i.e., the total number of terms multiplied by the average of each term.

We have now shown that both the lef-hand and right-hand sides equal $\frac{n(n-1)}{2}$, and thus must be equal. □

Problem 5. Prove Pascal's Identity: i.e., for $0 < k < n$ that:

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) + \left(\begin{array}{c} n-1 \\ k \end{array} \right)$$

Proof.

We can simplify the equality as using the defintion of each binomial coefficient as follows:

$$\frac{n!}{(n-k)!k!} = \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-k-1)!k!}$$

To further simplify, we want to put all fractions under a common denominator (i.e., $(n-k)!k!$).

The lefthand side is already in the common denominator.

The first summand of the right side needs to be multiplied by $\frac{k}{k} = 1$ because $k! = k(k-1)!$.

The second summand on the right side can be multiplied by $\frac{(n-k)}{(n-k)} = 1$ because $(n-k)! = (n-k)(n-k-1)!$.

Thus we get:

$$\frac{n!}{(n-k)!k!} = \frac{k(n-1)!}{(n-k)!k!} + \frac{(n-k)(n-1)!}{(n-k)!k!}$$

Simplifying:

$$\frac{n!}{(n-k)!k!} = \frac{k(n-1)! + (n-k)(n-1)!}{(n-k)!k!}$$

Note, the right-hand sides numerator can be rewritten as $(k + n - k)(n-1)! = n!$.

Thus, we can simplify further

$$\frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!k!}$$

We have thus proven the two sides equal. □

Problem 6. Prove for $0 \leq k \leq n$ that:

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n \\ n-k \end{array} \right)$$

Proof.

By can simplify the equation by substituting the definition for each binomial coefficient as follows:

$$\frac{n!}{(n-k)!k!} = \frac{n!}{(n-(n-k))!(n-k)!}$$

Note the denominator on the right-hand side can be simplifed (i.e., $(n-(n-k))! = k!$):

$$\frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!}$$

We have thus proven the equality. □