Contradiction: Group Exercises

CSCI 246

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Problem 1. Rewrite each of the statements into thier contra-positive.

A. If an integer n is divisible by 6, then n is divisible by 3.

If an integer n is not divisible by 3, then n is not divisible by 6.

B. If a function f is injective, then $f(x) = f(y)$ implies $x = y$.

For any function f, if $f(x) = f(y)$ and $x \neq y$, then f is not injective.

C. An integer n is even if and only if n^2 is even.

An integer n is odd if and only if n^2 is odd.

D. If $x|y$ and $y|z$, then $x|z$.

If $\neg(x|z)$, then $\neg(x|y)$ or $\neg(y|z)$.

E. A function f is a bijection if and only if f is injective and f is surjective.

A function f is not a bijection if and only if f is not injective or f is not surjective.

Problem 2. Prove that if n^2 is divisible by 25, then n is divisible by 5.

Proof.

Proof by contrapositive: If n is not divisible by 5, then n^2 is not divisible by 25.
By assumption 5 does not divide n; i.e., there is no k such that $n = 5k$.
Or alternatively, $n = 5k + c$ for some integers k and c in which $0 < c < 5$.
By squaring both sides we get $n^2 = 25k^2 + 10kc + c^2$.
Simplifying, we get $n^2 = 5(5k^2 + 2kc) + c^2$.
Necessarily, n^2 is divisible by 5 if and only if c^2 is divisible by 5.
However, since $0 < c < 5$, c^2 is either 1, 4, 9, or 16. Therefore, c^2 is not divisible by 5.
Thus, n^2 is not divisible by 5.
Assume that n^2 is divisble by 25; i.e., $n^2 = 25k'$ for some k'.
Necessarily $n^2 = 5(5k')$ thus 5 divides n^2.
A contradiction. We may conclude that 25 does not divide n^2. □

Problem 3. Prove that if ab is divisble by 7, then a is divisble by 7 or b is divisible by 7.

Proof.
Proof by contrapositive: if 7 does not divide a and 7 does not divide b, then 7 does not divide ab.
By assumption 7 does not divide a and 7 does not divide b.
Thus, there is no integer k_a such that $a = 7k_a$ and similarly no integer k_b such that $b = 7k_b$.
Or alternatively, there are some integers k_a, k_b, c_a and c_b such that:
$a = 7k_a + c_a$ and $b = 7k_b + c_b$ with $0 < c_a, c_b < 7$.
Necessarily, $ab = (7k_a + c_a)(7k_b + c_b) = 49k_ak_b + 7k_ac_b + 7k_bc_a + c_ac_b = 7(7k_ak_b + k_ac_b + k_bc_a) + c_ac_b$.
Clearly, ab is divisible by 7 if and only if c_ac_b is divisible by 7.
Since $c_a, c_b \in \{1, 2, 3, 4, 5, 6\}$, we have $c_ac_b \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36\}$.
Clearly, for any value of c_ac_b, 7 does not divide c_ac_b. Thus, 7 does not divide ab. □

Problem 4. Prove that there is no smallest rational number.

Proof.
Assume not: there is a smallest rational number $0 < \frac{a}{b}$.
That is, there is no $0 < \frac{c}{d}$ such that $\frac{c}{d} < \frac{a}{b}$.
Consider $\frac{a}{2b}$. Necessarily, $0 < \frac{a}{2b}$ and $\frac{a}{2b} < \frac{a}{b}$.
A contradiction. Thus there is no smallest rational number. □

Problem 5. Prove that if we choose 6 integers from $\{1, 2, 3, 4, 5\}$, then at least two must be equal.

Proof.
Assume not. There are 6 distinct values $a, b, c, d, e, f \in \{1, 2, 3, 4, 5\}$.
Consider the set $A = \{a, b, c, d, e, f\}$. Since all of $a, b, c, d, e, f \in \{1, 2, 3, 4, 5\}$, we have $A \subseteq \{1, 2, 3, 4, 5\}$.
Thus, $|A| \leq |\{1, 2, 3, 4, 5\}| = 5$.
Since all values a, b, c, d, e, f are distinct we know that $|A| = 6$.
A contradiction. Thus a, b, c, d, e, f cannot be distinct; i.e., at least two of a, b, c, d, e, f must be equal. □

Problem 6. Prove that for $A \subseteq \mathbb{Z}$, if A has a smallest element, then it is unique.

Proof. Let A be a subset of the integers.
Assume there are two distinct values $x, y \in A$ that are the smallest element of A.
That is, $\forall z \in A.\ x \leq z$ and similarly $\forall z \in A.\ y \leq z$.
Since x and y are in A, we know that $x \leq y$ and $y \leq x$. Thus $x = y$.
A contradiction, thus there must be at most 1 smallest element of A. □